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Continuous Dependence for Certain Degenerate Parabolic Equations*

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Consider time-periodic solutions of $\dot{u}-\nabla\cdot\gamma(|\nabla u|)\nabla u=f$, where γ may have an initial interval of degeneracy: $\gamma(r)=0$ for $0\leqslant r\leqslant r_0$. It is shown that, although u need not even be unique, one has continuous dependence on the data and also on the nonlinearity $\gamma(\cdot)$ for $\eta(\cdot):=\gamma(|\nabla u|)\nabla u$. This generalizes the results of T. I. Seidman (*J. Differential Equations* **19** (1975), 242–257) for the nondegenerate case.

1. INTRODUCTION

Let $\Omega\subset\mathbb{R}^d$ be a bounded spatial region (with suitably smooth boundary $\partial\Omega$). We will consider the problem of finding time-periodic solutions u for a time-periodic parabolic problem

$$\dot{u}=\nabla\cdot\gamma(\cdot,|\nabla u|)\nabla u+f \tag{1.1}$$

which we write in the form

$$\dot{u}-\nabla\cdot\eta=f \qquad \text{on} \qquad \mathcal{Q}:=\mathbb{P}\times\Omega$$

where

$$\eta:=\gamma(\cdot,|\xi|)\xi, \qquad \xi:=\nabla u. \tag{1.2}$$

Note that ξ, η are functions on $\mathcal{Q}$ taking values in $\mathbb{R}^d$. Here $\mathbb{P}$ denotes the period interval $[0,T]$ viewed as $\mathbb{R}/T\mathbb{Z}$, so that in viewing functions as defined on $\mathcal{Q}$ we are automatically imposing the requirement of time-periodicity with period T. We consider (1.2) with first order boundary conditions of the form

$$-\eta\cdot\mathbf{n}=\phi \qquad \text{on} \qquad \Sigma:=\mathbb{P}\times\partial\Omega. \tag{1.3}$$

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We note immediately that the consistency condition

$$\int_{\mathscr{Q}} f + \int_{\Sigma} \phi = 0 \tag{1.4}$$

is necessary for the existence of a solution to (1.2), (1.3).

One physical interpretation of (1.1), noted already in [6], relates to the analysis of eddy currents induced in a nonlinearly ferromagnetic conductor by an external magnetic field. Under conditions of longitudinal invariance the vector potential has a single nonvanishing component (e.g., the z-component, with Ω now a cross-sectional region in the x, y-plane) and, with some manipulation appropriate to the special situation, one can show that this component u satisfies the equation (1.1) with $f = 0$.

Here γ is the reciprocal of the magnetic permeability of the material. For certain cases one can take γ to be constant so the problem becomes linear; in the general case, however, γ is dependent on the strength of the magnetic field. Thus $\gamma = \gamma(|\mathbf{B}|)$ with $\mathbf{B} = \nabla_3 \times (0, 0, u)$ giving $|\mathbf{B}| = |\nabla_2 u|$ (subscripts on ∇ here indicate the relevant dimensionality). For an inhomogeneous material γ will also depend on position $x \in \Omega$ (and conceivably on time as well—in which case we must assume suitable periodicity for this explicit dependence). Note that $\xi = \nabla_2 u$ is to correspond to $\mathbf{B} = \nabla \times (0, 0, u)$ so $\mathbf{B} = (-u_y, u_x, 0) = (-\xi_2, \xi_1, 0)$ and $\eta \cdot \mathbf{n} = \gamma[u_x n_1 + u_y n_2] = \gamma \mathbf{B} \cdot \mathbf{t}$ with $t_1 = -n_2$, $t_2 = n_1$, $t_3 = 0$ so $\eta \cdot \mathbf{n}$ just gives the tangential part of $\mathbf{H} = \gamma \mathbf{B}$; this is the physically correct boundary condition. The induced eddy current is then given as $\mathbf{i} = \nabla \times \mathbf{H}$.

For data f, ϕ periodic in time we seek a solution with the same periodicity. The standard well-posedness result would be to show that the solution u exists uniquely and depends continuously on the pair $[f, \phi]$. This must, of course, be modified: we have already noted (1.4) and can also see immediately that u is, at best, determined only to within an additive constant. Our primary interest will actually be in the determination of η, rather than of u itself. Further, since the precise functional form of $\gamma(\cdot\cdot)$ is known only through measurement, the same logic as that of [6] suggests the need to have continuous dependence on γ as well as on $[f, \phi]$.

The present paper is intended as a generalization of [6], which presented just such continuous dependence results for $\gamma: \mathbb{R}^+ \to \mathbb{R}^+$ (i.e., no dependence on t, x) with $\gamma(r) > 0$ for $r > 0$ and, of course, other asymptotic conditions. Here, we wish to treat a degenerate case, admitting γ with an interval of degeneracy:

$$\gamma(r) = 0 \text{ for } 0 \leqslant r \leqslant r_0, \qquad \gamma(r) > 0 \text{ for } r > r_0. \tag{1.5}$$

We will also admit inhomogeneity but will restrict attention to γ "essentially constant" at ∞. This last is entirely for simplicity and the reader is invited to formulate the corresponding L^p theory for $\gamma \sim r^{p-2}$ along similar lines, (cf. [6, 7]).

The most striking aspect of the results to be presented is that we show continuity for the dependence of η on $[f, \phi, \gamma]$ but not for u. Note that in the motivating physical application it is the magnetic field **H** which is of real interest while u, itself, is only an instrumental construct: a component of the vector potential having no direct significance itself. Indeed, suppose one has $f=0$, $\phi=0$ in (1.2), (1.3) with γ as in (1.5) and let $\tilde{u}$ be *any* function defined on Ω (independent of t) with

$$|\nabla \tilde{u}| \leqslant r_0 \qquad \text{a.e. on } \Omega.$$

Then $\dot{\tilde{u}}=0$ and $\tilde{\eta} := \gamma(|\nabla \tilde{u}|)\nabla \tilde{u}=0$ by (1.4) so $\tilde{u}$ is a solution (trivially periodic) of (1.2), (1.3). Thus, we must accept an essential nonuniqueness for u. Despite this, we will see that "continuous dependence" considerations apply to $\eta := \gamma \nabla u$—which, after all, is equivalent to **H**, the item of primary physical significance.

For the magnetic field problem is not clear whether there could be such a threshhold effect as (1.5) with $r_0 > 0$ for any physical materials. Indeed, the continuous dependence result obtained here itself suggests that such an effect (degeneracy) would be difficult to distinguish experimentally. The present treatment, however, was stimulated by consideration of the use of such a degenerate γ in numerical computation [2] for approximation in a context in which the "true" (physical) γ is at least known to be quite small on $[0, r_0]$.

2. Formulation

Suppose we set

$$\omega(t) := \int_\Omega u(t, \cdot) \tag{2.1}$$

for a solution u of (1.2), (1.3). Then

$$\dot{\omega}(t) = \int_\Omega \dot{u} = \int_\Omega [f + \nabla \cdot \eta] = \int_\Omega f + \int_{\partial\Omega} \phi \tag{2.2}$$

so, subject to (1.4), ω is well-defined on $\mathbb{P}$ to within an additive constant by solving an ordinary differential equation on $[0, T]$ with known data. Next, consider the map

$$A\colon \xi \mapsto u_0 \colon \mathfrak{D} \to H^1(\Omega) \tag{2.3}$$

defined by

$$\nabla u_0 = \xi, \qquad \int_\Omega u_0 = 0, \tag{2.4}$$

where $\mathfrak{D}$ is the range of ∇: $H^1(\Omega) \to L^2(\Omega \to \mathbb{R}^d)$; i.e., $\mathbf{A} = \nabla^{-1}$. The decomposition

$$u = \mathbf{A}\xi + \omega/|\Omega| \tag{2.5}$$

then permits us to reformulate (1.1), in terms of ξ as the unknown function, potentially ranging over the space $\mathfrak{D}$ of L^2 gradient vector fields on $\Omega \subset \mathbb{R}^d$. It is well-known (cf., e.g., [8]) that $\mathfrak{D}$ is a closed subspace of $L^2(\Omega \to \mathbb{R}^d)$ and that $\mathbf{A}$ is then a well-defined isomorphism. (Note that when Ω is simply connected $\mathscr{D}$ is determined simply by the standard consistency conditions $\partial\xi_j/\partial x_k = \partial\xi_k/\partial x_j$, taken in a distributional sense.)

Setting

$$\hat{f} := f - \dot{\omega}/|\Omega|, \tag{2.6}$$

one has from (2.2) that

$$\int_\Omega \hat{f} + \int_{\partial\Omega} \phi = 0 \qquad \text{for each } t \text{ in } \mathbb{P} \tag{2.7}$$

and, formally, (2.5) gives $\dot{u} = \mathbf{A}\dot{\xi} + \dot{\omega}/|\Omega|$ so (1.2), (1.3) become

$$\mathbf{A}\dot{\xi} - \nabla \cdot \eta = \hat{f}, \qquad -\eta \cdot \mathbf{n} = \phi \qquad \text{with} \quad \eta = \gamma(|\xi|)\xi. \tag{2.8}$$

The boundary condition in (2.8) does not make sense pointwise for $\xi \in \mathscr{D}$, i.e., merely assumed to have L^2 regularity. We interpret this weakly, multiplying (2.8) by a test function v (which we may assume has the form $v = \mathbf{A}\zeta$) to obtain

$$\langle \mathbf{A}\zeta, \mathbf{A}\dot{\xi} \rangle + \langle \zeta, \eta \rangle = \int_\Omega v\hat{f} + \int_{\partial\Omega} v\phi. \tag{2.9}$$

on applying the Divergence Theorem to $\langle \mathbf{A}\xi, -\nabla \cdot \eta \rangle$. Let us solve

$$-\Delta w = \hat{f}, \qquad \nabla w \cdot \mathbf{n} = \phi, \qquad \int_\Omega w = 0 \tag{2.10}$$

for each t (note that (2.7) is just sufficient to ensure that this is possible, and set $\psi := \nabla w$. We set

$$\mathscr{F}_0 := L^2(\mathbb{P} \to [H^1(\Omega)]^* \times H^{-1/2}(\partial\Omega)) = \mathscr{F}_1 \times \mathscr{F}_2,$$

where

$$\mathscr{F}_1 := L^2(\mathbb{P} \to [H^1(\Omega)]^*), \qquad \mathscr{F}_2 := L^2(\mathbb{P} \to H^{-1/2}(\partial\Omega)),$$
$$\mathfrak{X} := \{\nabla v : v \in L^2(\mathbb{P} \to H^1(\Omega))\} = L^2(\mathbb{P} \to \mathfrak{D}) \subset \mathfrak{X}_0 := L^2(\mathscr{Q} \to \mathbb{R}^d) \tag{2.11}$$

and note that $[f, \phi] \mapsto \psi$ is continuous from $\mathscr{F}_0$ to $\mathfrak{X}$:

LEMMA 1. *For any $[f, \phi] \in \mathscr{F}_0$ one can define w for a.e. $t \in \mathbb{P}$ by* (2.10) *with*

$$\hat{f} := f - \left[\int_\Omega f + \int_{\partial\Omega} \phi\right] \Big/ |\Omega| \tag{2.12}$$

to obtain $w \in L^2(\mathbb{P} \to H^1(\Omega))$ so $\psi := \nabla w$ is in $\mathfrak{X}$. The resulting linear map, $\mathscr{F}_0 \to \mathfrak{X}$ is continuous.

Proof. We have $1 \in L^\infty(\mathbb{P} \to H^1(\Omega))$ so $(\int_\Omega f) \in L^2(\mathbb{P})$ which we can obviously embed in $\mathscr{F}_1$; similarly $(\int_{\partial\Omega} \phi) \in \mathscr{F}_1$ so (2.12) gives $\hat{f} \in \mathscr{F}_1$ with (2.7). For each $t \in \mathbb{P}$ one has solvability of (2.10) and the map (independent of t),

$$[f(t), \phi(t)] \mapsto w(t) \mapsto \psi(t) \colon [H^1(\Omega)]^* \times H^{-1/2}(\partial\Omega) \to H^1(\Omega) \to \mathfrak{D}, \tag{2.13}$$

is continuous, giving the desired result. ∎

For test functions $v = \mathbf{A}\zeta$ and $\psi = \nabla w$ as above, one has

$$\begin{aligned}\langle \zeta, \psi \rangle &= \int_\Omega \nabla v \cdot \nabla w = \int_{\partial\Omega} v(\nabla w \cdot n) - \int_\Omega v \, \Delta w \\ &= \int_{\partial\Omega} v\phi + \int_\Omega v\hat{f}\end{aligned}$$

and we write (2.9) as

$$\langle \mathbf{A}\zeta, \mathbf{A}\dot{\xi} \rangle + \langle \zeta, \eta \rangle = \langle \zeta, \psi \rangle, \qquad \eta = \gamma(\cdot, |\xi|)\xi, \tag{2.14}$$

where ψ now subsumes the role of the data $[f, \phi]$.

At this point we introduce an unbounded linear operator $\mathbf{L}$ and a nonlinear operator $\mathbf{G}_*$ acting on $\mathfrak{X}$. We define $\mathbf{L} \colon \mathfrak{X} \supset \mathscr{D}(L) \to \mathfrak{X}^* = \mathfrak{X}$ by

$$\mathbf{L}\xi := \mathbf{A}^*(\mathbf{A}\dot{\xi}) \qquad \text{(pointwise a.e. in } t) \text{ for } \xi \in \mathscr{D}(\mathbf{L}), \tag{2.15}$$

where, as is standard, the domain $\mathscr{D}(\mathbf{L})$ is taken maximal:

$$\begin{aligned}\mathscr{D}(\mathbf{L}) &:= \{\xi \in \mathfrak{X} \colon \mathbf{L}\xi \text{ (defined distributionally) is in } \mathfrak{X}\} \\ &:= \{\nabla v \in L^2(\Omega) \colon \dot{v} \in L^2(\mathbb{P} \to [H^1(\Omega)]^*)\}.\end{aligned} \tag{2.16}$$

Note that $\mathbf{L}$ is then a densely defined, closed linear operator on $\mathfrak{X}$; further, $\mathbf{L}$ is skew-adjoint ($\mathbf{L}^* = -\mathbf{L}$: integration by parts in t, noting the periodicity implied by the definition of $\mathbb{P}$). One easily sees that one may solve the vector Poisson equation on Ω,

$$-\Delta\lambda = \dot{\xi}, \qquad \lambda \cdot u = 0 \text{ on } \partial\Omega, \qquad \lambda \in \mathfrak{D}, \tag{2.17}$$

pointwise in t to obtain $\lambda = \mathbf{L}\xi$ (for smooth $\xi \in \mathfrak{X}$) and this may be taken as an alternative definition/interpretation of $\mathbf{L}$. Next, define $\mathbf{G}$ on the larger space $\mathfrak{X}_0$ by

$$[\mathbf{G}\xi](\cdot) := \gamma(\cdot, |\xi(\cdot)|)\, \xi(\cdot) \tag{2.18}$$

pointwise on $\mathcal{Q}$. Under the hypotheses,

(H-1) $\gamma: \mathcal{Q} \times \mathbb{R}_+ \to \mathbb{R}_+$ is uniformly bounded ($0 \leqslant \gamma \leqslant \bar{\gamma}$ on $\mathcal{Q} \times \mathbb{R}_+$) and satisfies Carathéodory conditions: measurability on $\mathcal{Q}$ for all $r \in \mathbb{R}_+$ and continuity on $\mathbb{R}_+$ a.e. on $\mathcal{Q}$,

one easily sees, noting standard properties of Nemytsky operators, that $\mathbf{G}\xi \in \mathfrak{X}_0$ for $\xi \in \mathfrak{X}_0$ and that $\mathbf{G}: \mathfrak{X}_0 \to \mathfrak{X}_0^* = \mathfrak{X}_0$ is continuous, taking bounded sets to bounded sets. Note that $\mathbf{G}$ is the Fréchet derivative of the functional

$$\xi \mapsto \int_{\mathcal{Q}} \Phi(\cdot, |\xi(\cdot)|): \mathfrak{X}_0 \to \mathbb{R}_+$$

with

$$\Phi(\cdot, r) := \int_0^r s\gamma(\cdot, s)\, ds. \tag{2.19}$$

We are not so much interested in $\mathbf{G}: \mathfrak{X}_0 \to \mathfrak{X}_0$ as in the operator $\mathbf{G}_*: \mathfrak{X} \to \mathfrak{X}$ defined by considering $\eta := \mathbf{G}\xi$ (for $\xi \in \mathfrak{X}$) as an element of $\mathfrak{X}_0^*$ and restricting its domain to $\mathfrak{X}$ to obtain an element of $\mathfrak{X}^* = \mathfrak{X}$, denoted by $\mathbf{G}_*\xi$. Thus,

$$\mathbf{G}_*: \xi \mapsto (\mathbf{G}\xi|_{\mathfrak{X}}): \mathfrak{X} \to \mathfrak{X}^* = \mathfrak{X}. \tag{2.20}$$

We also impose on γ the further hypotheses:

(H-2) $r\gamma(\cdot, r)$ is nondecreasing in r a.e. on $\mathcal{Q}$;

(H-3) $\gamma(\cdot, r) \geqslant \gamma - a_0(\cdot)/r^2$ with $\gamma > 0$ and $a_0 \in L^1(\mathcal{Q})$.

LEMMA 2. *Let* $\gamma: \mathcal{Q} \times \mathbb{R}_+ \to \mathbb{R}_+$ *satisfy* (H) = (H-1, 2, 3). *Then* $\mathbf{G}$: $\mathfrak{X}_0 \to \mathfrak{X}_0$ *and* $\mathbf{G}_*: \mathfrak{X} \to \mathfrak{X}$ *are continuous, monotone, coercive operators.*

Proof. Continuity has already been noted. Note that $\Phi(\cdot, r)$ is nondecreasing in r and (H-2) ensures convexity on $\mathbb{R}^d$ of $\xi \mapsto \Phi(\cdot, |\xi|)$ a.e. on $\mathcal{Q}$ so (2.19) gives a convex functional on $\mathfrak{X}_0$; the Fréchet derivative $\mathbf{G}$ is therefore monotone. From (H-3) a simple computation gives

$$\begin{aligned}\langle \mathbf{G}\xi, \xi\rangle &= \int_{\mathcal{Q}} r^2\gamma(\cdot, r) \qquad (r := |\xi(\cdot)|)\\ &\geqslant \gamma\|\xi\|^2 - \alpha_0 \qquad \left(:= \int_{\mathcal{Q}} a_0\right)\end{aligned} \tag{2.21}$$

so $\mathbf{G}$ is coercive. (*Remark*: The condition (H-3) for coercivity is sharp—compare Proposition 2.14 of [3].) Since

$$\langle \mathbf{G}_*\xi - \mathbf{G}_*\xi', \xi - \xi'\rangle = \langle \mathbf{G}\xi - \mathbf{G}\xi', \xi - \xi'\rangle \qquad \text{for} \quad \xi, \xi' \in \mathfrak{X},$$

the same monotonicity and coercivity properties also apply to $\mathbf{G}_*: \mathfrak{X} \to \mathfrak{X}$. ▮

We can now present the weak formulation of (1.1), which we will use:

> Given $[f, \phi] \in \mathscr{F} := \{[f, \phi] \in \mathscr{F}_0 : (1.4)\}$, define $\psi \in \mathfrak{X}$ as in Lemma 1. Then (1.2), (1.3) imply $\xi := \nabla u$ satisfies (2.11) for arbitrary $\xi \in \mathfrak{X}$ so, as an equation in $\mathfrak{X}^* = \mathfrak{X}$,
>
> $$\xi \in \mathscr{D}(\mathbf{L}), \qquad \mathbf{L}\xi + G_*\xi = \psi. \tag{2.22}$$

3. Existence, Etc.

It is (2.22) which we take as the appropriate weak formulation of (1.2), (1.3): given a solution ξ of (2.22) we can recover from it a solution u of (1.2), (1.3) by (2.5) with ω obtained by solving the ordinary differential equation (2.2), which is possible (uniquely to within an additive constant) by (1.4).

THEOREM 1. *Assume $\gamma: \mathcal{Q} \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the hypotheses* (H). *Then* (2.22) *has a solution $\xi \in \mathscr{D}(\mathbf{L}) \subset \mathfrak{X}$ for each $\psi \in \mathfrak{X}^* = \mathfrak{X}$.*

Proof. As noted following (2.15), (2.16), the linear operator L is skew-adjoint. By Lemma 2, $\mathbf{G}_*$ is continuous, monotone, coercive. Thus, by a theorem of Browder's (cf., e.g., [4]) the operator $(\mathbf{L} + \mathbf{G}_*)$ is maximal monotone and surjective, giving solvability of (2.22) for each $\psi \in \mathfrak{X}$. ▮

These arguments generalize immediately to some extent: we may consider replacing the identification $\eta := \gamma(\cdot, |\xi|)\xi$ by $\eta := g(\cdot, \xi)$, still with

$\xi := \nabla u$, where $g: \mathscr{Q} \times \mathbb{R}^d \to \mathbb{R}^d$ is more general than $g(\cdot, |\xi|)\xi$. We replace (H) by (H′):

(H′-1) $g: \mathscr{Q} \times \mathbb{R}^d \to \mathbb{R}^d$ is of linear growth: $|g(\cdot, \xi)| \leqslant b(\cdot) + \bar{\gamma}|\xi|$ with $b \in L^2(\mathscr{Q})$, and satisfies Carathéodory conditions;

(H′-2) a.e. on $\mathscr{Q}$, $g(\xi) = g(\cdot, \xi)$ satisfies: for $\xi, \xi' \in \mathbb{R}^d$, either $g(\xi) = g(\xi')$ or $[g(\xi) - g(\xi')] \cdot [\xi - \xi'] > 0$;

(H′-3) $g(\cdot, \xi) \cdot \xi \geqslant \underline{\gamma} |\xi|^2 = a_0$ with $\gamma > 0$ and $a_0 \in L^1(\mathscr{Q})$.

Note that (H′-2) in slightly stronger than just monotonicity but holds in the context of $g(\xi) := \gamma(|\xi|)\xi$ with (H-2).

LEMMA 3. *Let $g = \mathbb{R}^d \to \mathbb{R}^d$ be the gradient of a C^1 convex functional $\Phi: \mathbb{R}^1 \to \mathbb{R}$. Then* (H′-2) *holds.*

Proof. See the Appendix. ∎

If we now define $\mathbf{L}$ and $\mathbf{G}_*$ as in (2.12) and (2.16) but now with $\mathbf{G}$ given by

$$[\mathbf{G}\xi](\cdot) := g(\cdot, \xi(\cdot)), \tag{3.1}$$

noting that (H′-1) implies $\mathbf{G}: \mathfrak{X}_0 \to \mathfrak{X}_0$ for this $\mathbf{G}$, then the same argument as earlier gives the equivalence, through (2.5), of the weak formulation (2.22) and the equation

$$\dot{u} - \nabla \cdot g(\cdot, \nabla u) = f \qquad \text{on } \mathscr{Q} \text{ with (1.3) (subject to (1.4)).} \tag{3.2}$$

Note that, as in Lemma 1, we obtain monotonicity and coercivity of $\mathbf{G}$ from (H′-2) and (H′-3), respectively. We take this opportunity to adjoin a uniqueness result.

THEOREM 2. *Let $g: \mathscr{Q} \times \mathbb{R}^d \to \mathbb{R}^d$ satisfy* (H′). *Then* (2.22) *has a solution $\xi \in \mathscr{D}(\mathbf{L}) \subset \mathfrak{X}$ for each $\psi \in \mathfrak{X}$ with the uniqueness property: if ξ, ξ' are two solutions of* (2.19), *then $\mathbf{G}\xi = \mathbf{G}\xi'$, i.e., $g(\cdot, \xi(\cdot)) = g(\cdot, \xi'(\cdot))$ a.e. on $\mathscr{Q}$.*

Proof. The proof is exactly as for Theorem 1 except for the uniqueness property. If ξ, ξ' satisfy (2.22) with the same ψ, then subtracting gives $\mathbf{L}(\xi - \xi') + [\mathbf{G}_*\xi - \mathbf{G}_*\xi'] = 0$. Applying this to $(\xi - \xi') \in \mathfrak{X}$ (so $\mathbf{G}_*$ coincides with $\mathbf{G}$) and noting the skew-adjointness of $\mathbf{L}$, we have

$$0 = \langle \mathbf{G}\xi - \mathbf{G}\xi', \xi - \xi' \rangle = \int_{\mathscr{Q}} [g(\cdot, \xi) - g(\cdot, \xi')] \cdot [\xi - \xi'].$$

By the monotonicity of $\xi \mapsto g(\cdot, \xi)$ on $\mathbb{R}^d$, the integrand must vanish a.e. on $\mathscr{Q}$ so (H′-2) gives $g(\cdot, \xi) = g(\cdot, \xi')$ a.e. on $\mathscr{Q}$ as desired. ∎

While our principal concern is with $\eta := \mathbf{G}\xi = g(\cdot, \nabla u)$ rather than with u itself, we do note a mild regularity result for the solution u of (3.3): we show

$$u \in L^2(\mathbb{P} \to H^1(\Omega)) \cap H^1(\mathbb{P} \to [H^1(\Omega)]^*) \cap C(\mathbb{P} \to L^2(\Omega)). \tag{3.3}$$

The first is immediate since $\xi = \nabla u$ is in $L^2(\mathscr{Q} \to \mathbb{R}^d)$. The second comes from the fact that $\xi \in \mathscr{D}(\mathbf{L})$ gives $\dot{u} \in L^2(\mathbb{P} \to [H^1(\Omega)]^*)$. We are concerned, then, with obtaining a uniform bound on the $L^2(\Omega)$-norm of $u(t)$ whence a density argument gives continuity of $u(\cdot)$ on $\mathbb{P}$. Since ξ satisfying (2.22) $\langle \mathbf{G}\xi, \xi \rangle = \langle \psi, \xi \rangle$, the coercivity obtained from (H′-3) gives

$$\|\xi\| \leqslant [\|\psi\| + (\|\psi\|^2 + 4\gamma \|a_0\|_1)^{1/2}]/2\gamma \tag{3.4}$$

and so a corresponding bound for the $L^2(\mathbb{P} \to H_1(\Omega))$-norm of $v := \mathbf{A}\xi$ which, of course, dominates the $L^2(\mathscr{Q})$-norm. If, for example, we normalize so $\int \omega = 0$, then (2.2) gives $\omega \in H^1(\mathbb{P}) \subset C(\mathbb{P})$ with a bound. For some c, then, we know $\|u(s)\| \leqslant c$ for s in some subset of $\mathbb{P}$ with positive measure. Multiplying (1.4) by $2u$ and integrating over $(s, t) \times \Omega$ for such s gives

$$\|u(t)\|^2 + 2\int^t \int_\Omega \beta = \|u(s)\|^2 + 2\int_s^t \left[\int_\Omega fu + \int_{\partial\Omega} \phi u\right],$$

where $\beta := g(\cdot, \nabla u) \cdot \nabla u \geqslant 0$. Given $[f, \phi] \in \mathscr{F}_0$ and the available bound on $u \in L^2(\mathbb{P} \to H^1(\Omega))$, we can uniformly estimate the right hand side and so bound u in $L^\infty(\mathbb{P} \to L^2(\Omega))$, uniformly in t.

4. Well-Posedness

We will take the data for the problem

$$\begin{aligned} \dot{u} - v \cdot g(\cdot, \nabla u) &= f && \text{on } \mathscr{Q} \\ g(\cdot, \nabla u) \cdot \mathbf{n} &= \phi && \text{on } \Sigma \qquad \text{(periodicity in } t) \end{aligned} \tag{4.1}$$

to be $[g, f, \phi] \in \mathscr{G} \times \mathscr{F}$ where

$$\begin{aligned} \mathscr{G} &:= \{g : (H')], \\ \mathscr{F} &:= \{[f, \phi] \in \mathscr{F}_0 : (1.4)\}. \end{aligned} \tag{4.2}$$

For our present purposes the "output" of (4.1) will be $\eta := g(\cdot, \nabla u) \colon \mathscr{Q} \mapsto \mathbb{R}^d$. Since Lemma 1 gives continuity of the map:

$[f, \phi] \mapsto \psi : \mathscr{F} \to \mathfrak{X}$, we may think of the data as $[g, \psi] \in \mathscr{G} \times \mathfrak{X}$ and note that Theorem 2 just asserts that the map

$$\begin{gathered} [g, f, \phi] \mapsto [g, \psi] \mapsto \eta := \mathbf{G}\xi = g(\cdot, \nabla u) \\ \mathscr{G} \times \mathscr{F} \to \mathscr{G} \times \mathfrak{X} \to \mathfrak{X}_0 \end{gathered} \tag{4.3}$$

is well-defined by (2.2), corresponding to (4.1). The principal object of this paper is to show that (4.3) is continuous.

We have $\mathscr{F}$, $\mathfrak{X}$ topologized as closed subspaces of $\mathscr{F}_0$, $\mathfrak{X}_0$ but must impose a topology on $\mathscr{G}$ to consider continuity of (3.3):

DEFINITION. We say $g_k \to g$ in $\mathscr{G}$ if

$$\begin{aligned} &\text{(i)} \quad g, g_k \in \mathscr{G} \text{ with } \gamma, \bar{\gamma}, b, a_0 \text{ fixed in (H}'\text{-1, 3);} \\ &\text{(ii)} \quad \text{a.e. on } \mathscr{Q} \text{ one has } g_k(\cdot, \xi) \to g(\cdot, \xi) \text{ for every } \xi \in \mathbb{R}^d. \end{aligned} \tag{4.4}$$

LEMMA 4. *Let $g_k \to g$ in $\mathscr{G}$. Then $\mathbf{G}_k \xi \to \mathbf{G}\xi$ in $\mathfrak{X}_0$ for each fixed $\xi \in \mathfrak{X}_0$. Each $\mathbf{G}_k$ is defined from g_k as in* (3.2).

Proof. From (4.4ii) we have

$$[\mathbf{G}_k \xi](\cdot) := g_k(\cdot, \xi(\cdot)) \to g(\cdot, \xi(\cdot)) =: [\mathbf{G}\xi](\cdot) \text{ pointwise a.e. on } \mathscr{Q}.$$

From (4.4i) we have $|g_k(\cdot, \xi(\cdot))| \leqslant b(\cdot) + \bar{\gamma}|\xi(\cdot)| \in L^2(\mathscr{Q})$ so the Dominated Convergence Theorem gives $\|\mathbf{G}_k \xi\|^2 \to \|\mathbf{G}\xi\|^2$. For subsequences we have weak convergence $\mathbf{G}_k \xi \rightharpoonup \eta$ with, necessarily, $\eta = \mathbf{G}\xi$ so $\mathbf{G}_k \xi \rightharpoonup \mathbf{G}\xi$ for the full sequence by uniqueness of the limit. Thus, $\mathbf{G}_k \xi \to \mathbf{G}\xi$ strongly in $\mathfrak{X}_0$. ▮

LEMMA 5. *Let $g_k \to g$ in $\mathscr{G}$. Assume $\xi_k \rightharpoonup \xi$ and $\eta_k := \mathbf{G}_k \xi_k \rightharpoonup \eta$ (weak convergence in $\mathfrak{X}_0$) and suppose $\langle \eta_k, \xi_k \rangle \to \langle \eta, \xi \rangle$. Then $\eta = \mathbf{G}\xi$ and one has $\eta_k \to \eta$ (strong convergence in $\mathfrak{X}_0$).*

Proof. See the Appendix. ▮

Using this lemma, the proof of our continuity theorem is quite simple.

THEOREM 3. *Let $g_k \to g$ in $\mathscr{G}$ and $[f_k, \phi_k] \to [f, \phi]$ in $\mathscr{F}$ so $\psi_k \to \psi$ in $\mathfrak{X}$. Obtain $\xi := \nabla u$ from* (4.1) *or, equivalently, ξ satisfying* (2.22); *similarly obtain ξ_k* using *$[g_k, f_k, \phi_k]$ or $[g_k, \psi_k]$. Then $\eta_k := \mathbf{G}_k \xi_k \to \eta := \mathbf{G}\xi$ in $\mathfrak{X}$.*

Proof. Denote by $\mathbf{P}$ the projection: $\mathfrak{X}_0^* \to \mathfrak{X}^*$ (equivalently, $\mathfrak{X}_0 \to \mathfrak{X}$) defined by restriction; i.e., $\mathbf{P}\eta := \eta|_{\mathfrak{X}}$ for $\eta \in \mathfrak{X}_0^*$. Thus $\mathbf{G}_* \xi = \mathbf{P}(\mathbf{G}\xi)$, and we have

$$\mathbf{L}\xi_k + \mathbf{P}\eta_k = \psi_k \qquad (\eta_k := \mathbf{G}_k \xi_k). \tag{4.5}$$

Applying this to ξ_k gives, in view of the skew-adjointness of $\mathbf{L}$,

$$\langle \eta_k, \xi_k \rangle = \langle \psi_k, \xi_k \rangle \leqslant \|\psi_k\| \, \|\xi_k\|.$$

From (H′-3) we obtain, then, a bound on $\{\xi_k\}$ whence, using (H′-1), also a bound on $\{\eta_k\}$. Extracting a subsequence if necessary, we may assume $\xi_k \rightharpoonup \hat{\xi}$ and $\eta_k \rightharpoonup \hat{\eta}$ in $\mathfrak{X}$-weak. Since $\mathbf{P}$ is continuous, we have weak convergence $\mathbf{P}\eta_k \rightharpoonup \mathbf{P}\hat{\eta}$ so (4.5) gives $\mathbf{L}\xi_k \rightharpoonup \psi - \mathbf{P}\hat{\eta}$. As $\mathbf{L}$ is closed, we then have $\mathbf{L}\hat{\xi} = \psi - \mathbf{P}\hat{\eta}$. Now the strong convergence $\psi_k \to \psi$ gives

$$\langle \eta_k, \xi_k \rangle = \langle \psi_k, \xi_k \rangle \to \langle \psi, \hat{\xi} \rangle = \langle \mathbf{L}\hat{\xi} + \mathbf{P}\hat{\eta}, \hat{\xi} \rangle = \langle \hat{\eta}, \hat{\xi} \rangle.$$

Hence, again using the skew-adjointness of $\mathbf{L}$ we may apply Lemma 5 to assert strong convergence $\eta_k \to \hat{\eta} = \mathbf{G}\hat{\xi}$. As we already know $\mathbf{L}\hat{\xi} + \mathbf{P}\hat{\eta} = \psi$, which shows that $\hat{\xi}$ is a solution of (2.22), and we may apply the uniqueness part of Theorem 2 to see that $\hat{\eta} = \eta := \mathbf{G}\xi$. The uniqueness of the limit shows that $\eta_k \to \eta$ without extracting subsequences. ∎

5. An Estimate

The original version of this material was split off from [7] and so estimates of the nature developed there were used to obtain the continuous dependence result in the context of (1.2), (1.3) using the formulation through (2.22). We are indebted to P. Benilan for pointing out [1] that a somewhat more general result could be obtained along the lines presented above, using techniques (see the Appendix) which are fairly standard in "monotone operator theory."

It might be of some interest, however, to see how the techniques of [7] could be used to obtain a somewhat more explicit estimate of the difference $(\eta - \eta')$ when η is obtained from

$$\eta := g(\cdot, \xi(\cdot)), \xi \in \mathscr{D}(\mathbf{L}), \qquad \mathbf{L}\xi + \mathbf{P}\eta = \psi \tag{5.1}$$

and correspondingly,

$$\eta' := g'(\cdot, \xi'(\cdot)), \ \xi' \in \mathscr{D}(\mathbf{L}), \qquad \mathbf{L}\xi' + \mathbf{P}\eta' = \psi'. \tag{5.2}$$

Here, of course, we have data $g, g' \in \mathscr{G}$ and $\psi, \psi' \in \mathfrak{X}$ with $\mathbf{L}$, $\mathbf{P}$ as above. We assume that g, g' each satisfy (H′) with parameters as given so we have available a bound on $|\eta|$, $|\eta'|$ from the coercivity: use (H′-3) to bound $|\xi|$ as in (3.4) and then (H′-i) to bound $|\eta|$; similarly for $|\eta'|$. We also assume we can estimate

$$\begin{aligned} |g(\cdot, \xi') - g'(\cdot, \xi')| = |\mathbf{G}\xi' - \mathbf{G}\xi'| &\leqslant \varepsilon_0, \\ |\psi - \psi'| &\leqslant \varepsilon_1. \end{aligned} \tag{5.3}$$

To make an explicit estimate possible we must introduce a more quantitative version of (H′-2), pointwise on $\mathscr{Q}$. Thus, we assume we have a function $\mu: \mathscr{Q} \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\begin{gathered} \mu \text{ is nonincreasing in } r \text{ and} \\ |\eta_1 - \eta_2|^2 \leqslant \mu(\cdot, r)\beta \qquad (r := \max\{|\eta_1|, |\eta_2|\}) \end{gathered} \tag{5.4}$$

where

$$\eta_j := g(\cdot, \xi_j) \text{ for arbitrary } \xi_j \in \mathbb{R}^d \qquad \text{and} \qquad \beta := (\eta_1 - \eta_2) \cdot (\xi_1 - \xi_2).$$

We then set

$$\begin{aligned} m(\cdot, \lambda) &:= \sup\{r: \mu(\cdot, r) > \lambda\}, \\ M(\lambda) &:= \int_{\mathscr{Q}} m^2(\cdot, \lambda). \end{aligned} \tag{5.5}$$

Observe that, simply because $\mu(\cdot, r)$ is nonincreasing in r, we must have $m(\cdot, \lambda) < \infty$ for large enough λ with $m(\cdot, \lambda) \to 0$ (pointwise a.e.) as $\lambda \to \infty$. Indeed, the Monotone Convergence Theorem gives $M(\lambda) \to 0$ as $\lambda \to \infty$ provided we assume that

$$M(\bar{\lambda}) < \infty \qquad \text{for some } \bar{\lambda}. \tag{5.6}$$

We can then define

$$\sigma(B) := \inf_\lambda\{[\lambda\beta + 4M(\lambda)]^{1/2}\} \tag{5.7}$$

and note that

$$\sigma: \mathbb{R}_+ \to \mathbb{R}_+ \text{ is well-defined and nondecreasing with } \sigma(B) \to 0 \text{ as } B \to 0. \tag{5.8}$$

LEMMA 6. *Let $g, g' \in \mathscr{G}$ and $\psi, \psi' \in \mathfrak{X}$; obtain ξ, η, ξ', η' as in* (5.1), (5.2). *Assume we have estimates* (5.3) *and a bound $|\xi - \xi'| \leqslant R$ (e.g., from coercivity, bounding $|\xi| + |\xi'|$). Let g be such that* (5.4), (5.6) *hold. Then*

$$|\eta - \eta'| < \varepsilon_0 + \sigma(R[\varepsilon_0 + \varepsilon_1]). \tag{5.9}$$

Proof. Pointwise, set $\hat{\eta}(\cdot) := g(\cdot, \xi'(\cdot))$ and $\beta(\cdot) := [\eta(\cdot) - \hat{\eta}(\cdot)] \cdot [\xi(\cdot) - \xi'(\cdot)]$. For any λ (presumably such that $M(\lambda) < \infty$), let $\mathscr{U} = \mathscr{U}_\lambda$ be the subset of $\mathscr{Q}$ on which $|\eta(\cdot)|, |\hat{\eta}(\cdot)| \leqslant m(\cdot, \lambda)$ and let $\mathscr{V} = \mathscr{V}_\lambda$ be its complement $\mathscr{Q}\backslash\mathscr{U}$. Note that on $\mathscr{V}$ one has $r = r(\cdot) := \max\{|\hat{\eta}(\cdot)|, |\eta(\cdot)|\} \geqslant m(\cdot, \lambda)$ so $\mu(\cdot, r) \leqslant \lambda$. Hence

$$\begin{aligned} |\eta(\cdot) - \hat{\eta}(\cdot)|^2 &\leqslant \lambda\beta(\cdot) && \text{on } \mathscr{V} = \mathscr{V}_\lambda \text{ by (5.4)}, \\ |\eta(\cdot) - \hat{\eta}(\cdot)|^2 &\leqslant 4m^2(\cdot, \lambda) && \text{on } \mathscr{U} = \mathscr{U}_\lambda. \end{aligned} \tag{5.10}$$

We have, then

$$\begin{aligned} |\eta - \hat{\eta}|^2 &= \int_{\mathscr{U}} |\eta(\cdot) - \hat{\eta}(\cdot)|^2 + \int_{\mathscr{V}} |\eta(\cdot) - \hat{\eta}(\cdot)|^2 \\ &\leqslant 4 \int_{\mathscr{U}} m^2(\cdot, \lambda) + \lambda \int_{\mathscr{V}} \beta(\cdot) \leqslant 4M(\lambda) + \lambda B, \end{aligned}$$

where

$$B := \int_{\mathscr{Q}} \beta(\cdot) = \langle \eta - \hat{\eta}, \xi - \hat{\xi} \rangle.$$

Since λ is arbitrary, this gives

$$|\eta - \eta'| \leqslant \sigma(B). \tag{5.11}$$

Now, taking the difference of (5.1), (5.2) applied to $(\xi - \xi') \in \mathfrak{X}$ and noting the skew-adjointness of $\mathbf{L}$, we obtain

$$B = \langle (\psi - \psi') + (\hat{\eta} - \eta'), \xi - \xi' \rangle \leqslant [\varepsilon_1 + \varepsilon_1] R. \tag{5.12}$$

Since σ is nondecreasing, this gives

$$|\eta - \eta'| \leqslant |\eta' - \hat{\eta}| + |\eta - \hat{\eta}| \leqslant \varepsilon_0 + \sigma(R[\varepsilon_1 + \varepsilon_0])$$

as in (5.9). ∎

To complete this line of argument we indicate how to obtain (5.4) in the original context.

LEMMA 7. *Suppose g has the form $g(\xi) := \gamma(|\xi|)\xi$ for $\xi \in \mathbb{R}^d$ and set $\bar{g}(u) := u\gamma(u)$ so $|g(\xi)| = \bar{g}(|\xi|)$. Assume g is nondecreasing and, for $r > 0$, set*

$$\mu(r) := 9 \sup\{d\bar{g}(u)/du \colon u > r/4\}. \tag{5.13}$$

Then, for any ξ, $\xi' \in \mathbb{R}^d$ giving $\eta := g(\xi)$, $\eta' := g(\xi')$, one has

$$|\eta - \eta'|^2 \leqslant \mu(r)[\eta - \eta'] \cdot [\xi - \xi'] =: \mu(r)\beta, \tag{5.14}$$

where $r := \max(|\eta|, |\eta'|)$.

Proof. With no loss of generality, suppose $r=|\eta|\geqslant|\eta'|=:s$ corresponding to $u:=|\xi|\geqslant|\xi'|=:v$. With $|\theta|\leqslant 1$ we have $\xi\cdot\xi'=\theta uv$, $\eta\cdot\eta'=\theta rs$ and, by direct computation, we obtain the identities

$$\begin{aligned} |\eta-\eta'|^2&=(r-\theta s)(r-s)+(1-\theta)(r+s)s,\\ \beta:=(\eta-\eta')\cdot(\xi-\xi')&=(r-\theta s)(u-v)+(1-\theta)(r+s)v. \end{aligned} \tag{5.15}$$

Case 1. $[s\leqslant r/2]$: We have $(r-\theta s)\geqslant r/2$ and, with $g(\hat{u})=r/2$, we have $r-r/2\leqslant[\inf\bar{g}'](u-\hat{u})\in[\mu(r)/9](u-v)$ as monotonicity of $\bar{g}$ gives $v\leqslant u$ for $s\leqslant r/2$. From (5.15), these combine to give

$$\frac{r^2}{4}\leqslant\frac{\mu(r)}{9}(r-\theta s)(u-v)\leqslant\frac{\mu(r)}{9}\beta$$

and so (5.14) since $s\leqslant r/2$ gives $|\eta-\eta'|\leqslant r+s\leqslant 3r/2$.

Case 2. $[s\geqslant r/2]$: We have $r-s\leqslant[\mu(r)/9](u-v)$ and, with $g(\hat{v})=s/2$, we have $s/2=s-s/2\leqslant[\mu(r)/9](v-\hat{v})\in[\mu(r)/9]v$. Thus (5.15) gives

$$\begin{aligned} |\eta-\eta'|^2&\leqslant[\mu(r)/9][(r-\theta s)(u-v)+(1-\theta)(r+s)\,2v]\\ &\leqslant[2\mu(r)/9]\beta, \end{aligned}$$

and so we have (5.14) in this case as well. ∎

Applying Lemma 7 pointwise gives (5.4), the monotonicity of μ being obvious from (5.13). We remark that it is not really necessary that γ be differentiable: it is entirely adequate to work with the modified definition

$$\mu(r):=\sup\left\{\frac{\bar{g}(u)-\bar{g}(v)}{u-v}:u>v>0,\,u>r/4\right\} \tag{5.13$'$}$$

which is equivalent to (5.13) when γ is differentiable. We are, of course, implicitly assuming that μ, so defined, is finite for $r>0$ (a.e. on $\mathscr{Q}$) and must also assume (5.7) for applicability of Lemma 6.

Appendix

We are indebted to P. Benilan for the observation that one could use methods fairly standard in monotone analysis to simplify the original argument. Specifically, the results needed are Lemmas 3 and 5, above. For completeness we include proofs here.

Proof (of Lemma 3). The condition on g implies that is is *hemicontinuous* and *trimonotone*:

$$g(\xi_1)\cdot[\xi_1-\xi_2]+g(\xi_2)\cdot[\xi_2-\xi_3]+g(\xi_3)\cdot[\xi_3-\xi_1]\geqslant 0. \tag{A.1}$$

(We have $g(\xi_1)\cdot[\xi_1-\xi_2]+[\Phi(\xi_2)-\Phi(\xi_1)]\geqslant 0$, etc., because $[g(\xi_1), 1]$ is a support functional at $[\xi_1, \Phi(\xi_1)]$ to the convex epigraph of Φ.) Wherever $[g(\xi)-g(\xi')]\cdot[\xi-\xi']=0$, we set $\xi_1=\xi$, $\xi_2=\xi'$, $\xi_3=\xi+t\zeta$ (ζ arbitrary in $\mathbb{R}^d$, $t>0$) in (3.1) to obtain

$$\begin{aligned}0&\leqslant g(\xi)\cdot(\xi\xi')+g(\xi')\cdot(\xi'-\xi-t\zeta)+g(\xi+t\zeta)\cdot(t\zeta)\\&=t[g(\xi+t\zeta)-g(\xi')]\cdot\zeta.\end{aligned}$$

Dividing by t and then letting $t\to 0^+$ gives $[g(\xi)-g(\xi')]\cdot\zeta\geqslant 0$ for arbitrary $\xi\in\mathbb{R}^d$ so $g(\xi)=g(\xi')$. ∎

Proof (of Lemma 5). For any $\xi'\in\mathfrak{X}_0$ we have $\mathbf{G}_k\xi'\to\mathbf{G}\xi'$ by Lemma 4 so, by the hypotheses,

$$\langle\eta-\mathbf{G}\xi', \xi-\xi'\rangle=\lim\langle\mathbf{G}_k\xi_k-\mathbf{G}_k\xi', \xi_k-\xi'\rangle\geqslant 0.$$

This, for arbitrary ξ' gives $\eta=\mathbf{G}\xi$ since $\mathbf{G}$ is maximal monotone. Now set

$$\beta_k(\cdot):=[g_k(\cdot,\xi(\cdot))-g_k(\cdot,\xi_k(\cdot))]\cdot[\xi(\cdot)-\xi_k(\cdot)].$$

We have $\beta_k\geqslant 0$ by (H′-2) a.e. on $\mathcal{Q}$ so

$$\|\beta_k\|_1=\int_{\mathcal{Q}}\beta_k=\langle\mathbf{G}_k\xi-\eta_k, \xi-\xi_k\rangle\to 0$$

by the hypotheses and Lemma 4. Extracting a subsequence, e.g., so $\sum\|\beta_k\|_1<\infty$, we may assume $\beta_k(\cdot)\to 0$ pointwise a.e. on $\mathcal{Q}$ with $\beta_k\leqslant\bar\beta(\cdot):=\sum\beta_k\in L^1(\mathcal{Q})$. From (H′-1) we have (pointwise a.e. on $\mathcal{Q}$)

$$\underline{\gamma}\,|\xi_k|^2\leqslant\eta_k\cdot\xi_k+a_0=a_0+\beta_k-\eta\cdot\xi+\eta\cdot\xi_k+\eta_k\cdot\xi.$$

Using (H′-3) and $\beta_k\leqslant\bar\beta$, this gives

$$\underline{\gamma}\,|\xi_k|^2\leqslant c_0+c_1\,|\xi_k|$$

with

$$c_0:=a_0+\bar\beta-\eta\cdot\xi+b\,|\xi|,\qquad c_1:=|\eta|+\bar\gamma\,|\xi|$$

from which we have

$$|\xi_k|\leqslant\bar\xi:=[c_1+(c_1^2+4c_0)^{1/2}]/2\underline{\gamma}. \tag{A.2}$$

Note that $c_0 \in L^1(\mathscr{Q})$, $c_1 \in L^2(\mathscr{Q})$ so $\bar{\xi} \in L^2(\mathscr{Q})$. By (H′-3) we have, again a.e. on $\mathscr{Q}$,

$$|\eta_k| \leqslant \bar{\eta} := b + \bar{\gamma}\bar{\xi} \in L^2(\mathscr{Q}). \tag{A.3}$$

We now wish to show that $\eta_k \to \eta$ pointwise a.e. on $\mathscr{Q}$. Neglecting a set of measure zero, we have (pointwise) $|\xi_k| \leqslant \bar{\xi} < \infty$ so, by compactness in $\mathbb{R}^d$, one may find (still pointwise) a subsequence $\{k = k(j) = k(\cdot, j)\}$ for which $\xi_{k(j)}(\cdot) \to \hat{\xi} \in \mathbb{R}^d$ and, by (4.4ii), $\eta_{k(j)}(\cdot) \to g(\cdot, \hat{\xi})$. We have

$$0 = \lim \beta_{k(j)} = [g(\cdot, \xi(\cdot)) - g(\cdot, \hat{\xi})] \cdot [\xi(\cdot) - \hat{\xi}].$$

By (H′-2), it follows that $g(\cdot, \hat{\xi}) = g(\cdot, \xi(\cdot)) = \eta(\cdot)$. By the uniqueness of the limit, it follows that $\eta_k(\cdot) \to g(\cdot, \xi(\cdot))$ at the (almost arbitrary) point of $\mathscr{Q}$ without extracting subsequences, i.e., $\eta_k \to \eta$ pointwise a.e. on $\mathscr{Q}$. In view of the estimate (4.6), we then have strong convergence in $\mathfrak{X}_0$ as desired. ∎

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